

# Affine polar spaces derived from symplectic spaces, their geometry and representations: alternating semiforms

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## Abstract

Deleting a hyperplane from a polar space associated with a symplectic polarity we get a specific, symplectic, affine polar space. Similar geometry, called an affine semipolar space arises as a result of generalization of the notion of an alternating form to a semiform. Some properties of these two geometries are given and their automorphism groups are characterized.

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## Introduction

In [5] affine polar spaces are derived from polar spaces the same way as affine spaces are derived from projective spaces, i.e. by deleting a hyperplane from a polar space embedded into a projective space. So, affine polar spaces (aps'es, in short) are embeddable in affine spaces and this let us think of them as of suitable reducts of affine spaces.

In general we have two types of affine polar spaces. Structures of the first type are associated with polar spaces determined by sesquilinear forms; one can loosely say: these are "stereographical projections of quadrics". They can also be thought of as determined by sesquilinear forms defined on vector spaces which represent respective affine spaces. These structures and adjacency of their subspaces were studied in [16]. Contrary to [5], in this approach Minkowskian geometry is not excluded. In particular, the result of [16] generalizes Alexandrov-Zeeman theorems originally concerning adjacency *of points* of an affine polar space (cf. [1], [20]).

The second class of affine polar spaces, which is included in [5] but is excluded from [16], consists of structures associated with polar spaces determined by symplectic polarities. The aim of this paper is to present in some detail the geometry of the structures in this class from view of the affine space in which they are embedded. The position of the class of thus obtained structures – let us call them symplectic affine polar spaces – is in many points a particular one.

Firstly, symplectic affine polar spaces are associated with null-systems, quite well known polarities in projective spaces with all points selfconjugate. So, symplectic aps'es have famous parents. Moreover, in each even-dimensional pappian projective

space such a (projectively unique) polarity exists. Thus a symplectic aps is not an exceptional space, but conversely, it is also a “canonical” one in each admissible dimension.

A second argument refers to the position of the class of symplectic aps’es in the class of all aps’es. As an affine polar space is obtained by deleting a hyperplane from a polar space, while the latter is realized as a quadric in a metric projective space, the derived aps appears as a fragment of the derived affine space. If the underlying form that determines the polar space is symmetric then the corresponding aps can be realized *on* an affine space in one case only – when the deleted hyperplane is a tangent one. And then the affine space in question is constructed *not* as a reduct of the surrounding projective space but as a derived space as it is done in the context of chain geometry (cf. [8], [2]). Moreover, such an aps can be represented without the whole machinery of polar spaces: it is the structure of isotropic lines of a metric affine space.

The only case when the point set of the reduct of a polar space *is* the point set of an affine space arises when we start from a null system i.e. in the case considered in the paper. But such an aps is not associated with a metric affine space i.e. with a vector space endowed with a nondegenerate bilinear symmetric form. What is a natural analytic way in which a symplectic aps can be represented, when its point set is represented via a vector space? A way to do so is proposed in our paper: to this aim we consider a “metric”, a binary scalar-valued operation defined on vectors. It is not a metric, in particular, it is not symmetric, and it is not invariant under affine translations. Nevertheless, it suffices to characterize respective geometry.

Symplectic aps’es have famous parents but they have also remarkable relatives. Although the “metric”: the analytical characteristic invariant of symplectic aps’es is not a form, it is closely related to forms. Loosely speaking, it is a sum of an alternating form  $\eta$  defined on a subspace and an affine vector atlas defined on a vector complement of the domain of  $\eta$ . Immediate generalization with ‘an alternating map’ substituted in place of ‘an alternating form’ comes to mind. Such a definition of a map may seem artificial, the resulting maps, which we call *semiforms*, have quite nice synthetic characterization though. Their basic properties are established in Section 2. A symplectic ‘metric’ appears to be merely a special instance of such a general definition and many problems concerning it (so as to mention a characterization of the automorphism group) can be solved in this general setting easier. To illustrate and to motivate such a general definition we show in 2.4 a semiform associated with a vector product, that yields also an interesting geometry. On the other hand, this geometry has close connections (see 2.4-B) with a class of hyperbolic polar spaces.

A semiform induces an incidence geometry that we call an *affine semipolar space* (cf. (19) and (20)). It is a  $\Gamma$ -space with affine spaces as its singular subspaces (cf. 2.17), and with generalized null-systems comprised by lines and planes through a fixed point (cf. 2.25; comp. a class with similar properties considered in [6]). In the paper we do not go any deeper into details of neither geometry of semiforms nor geometries other than symplectic aps’es. We rather concentrate on “aps’es and around”.

Finally, we pass to our third group of arguments: that geometry of symplectic aps’es is interesting on its own right. Geometry of affine polar spaces is, by defini-

tion, an incidence geometry i.e. an aps is (as it was defined both in [5] and [16]) a partial linear space: a structure with points and lines. From the results of [4] we get that geometry of symplectic affine polar spaces can be also formulated in terms of binary collinearity of points – an analogue of the Alexandrov-Zeeman Theorem. A characterization of aps'es as suitable graphs is not known, though.

The affine polar spaces associated with metric affine spaces (as it was sketched above) can be, in a natural consequence, characterized in the “metric” language of line orthogonality or equidistance relation inherited from the underlying metric affine structure. It is impossible to investigate a line orthogonality imposed on an affine structure so as it gives rise to a symplectic aps. However, in case of a symplectic aps a “metric” mentioned above determines an “equidistance” relation which can be used as a primitive notion to characterize the geometry. There is no general commonly accepted axiom system of a weak equidistance relation (of a congruence of segments, in other words). A very natural one, that characterizes metric affine spaces is presented in [19]. Roughly speaking, in accordance with that approach a congruence of segments is an equivalence relation on pairs of points such that bisector hyperplanes are really affine hyperplanes. But these properties are met by our “symplectic equidistance” as well. The difference is that a segment and its translate need not be congruent under our equidistance. In this paper we do not intend to give a characterization of symplectic aps'es in the language of equidistance. Nevertheless, we think it is worth to stress on that this is also a possible language for this geometry and to indicate similarities and dissimilarities between our equidistance and that used in metric affine geometry. Since our equidistance is not commutative we have two types of bisectors (cf. (34)) and thus two types of symmetry under a hyperplane (cf. 3.14). The first type of a relation of being symmetric wrt. a hyperplane is related to translations and the other type is a central symmetry.

## 1 Definitions and preliminary results

Recall that the affine space  $\mathbf{A}(\mathbb{V})$  defined over a vector space  $\mathbb{V}$  has the vectors of  $\mathbb{V}$  as its points and the cosets of the 1-dimensional subspaces of  $\mathbb{V}$  as its lines.

We write  $\tau_\omega$  for the (affine) translation on the vector  $\omega$ ,  $\tau_\omega(x) = x + \omega$ .

### 1.1 Polar spaces

Let  $\mathbb{W}$  be a vector space over a (commutative) field  $\mathfrak{F}$  with characteristic  $\neq 2$  and let  $\xi$  be a nondegenerate bilinear reflexive form defined on  $\mathbb{W}$ . Assume that the form  $\xi$  has finite index  $m$  and  $n = \dim(\mathbb{W})$ . We will write  $\text{Sub}(\mathbb{W})$  for the class of all vector subspaces of  $\mathbb{W}$  and  $\text{Sub}_k(\mathbb{W})$  for the class of all  $k$ -dimensional subspaces. In the projective space  $\mathfrak{P} = \langle \text{Sub}_1(\mathbb{W}), \text{Sub}_2(\mathbb{W}), \subset \rangle$  the form  $\xi$  determines the polarity  $\delta = \delta_\xi$ . We write  $\mathbf{Q}(\xi)$  for the class of isotropic subspaces of  $\mathbb{W}$ :

$$\mathbf{Q}(\xi) = \{U \in \text{Sub}(\mathbb{W}) : \xi(U, U) = 0\}; \quad \mathbf{Q}_k(\xi) = \{U \in \mathbf{Q}(\xi) : \dim(U) = k\}.$$

Assume that  $m \geq 2$ . The structure

$$\mathbf{Q}_\xi(\mathbb{W}) := \langle \mathbf{Q}_1(\xi), \mathbf{Q}_2(\xi), \subset \rangle$$

is referred to as the *polar space* determined by  $\delta$  in  $\mathfrak{P}$ .

## 1.2 Hyperbolic polar spaces and their reducts

This section may look superfluous from view of symplectic polar spaces but it is used later in an example which justifies our general construction of semiforms.

Now let  $\xi$  be symmetric and  $\perp = \perp_\xi$  be the orthogonality determined by  $\xi$  on  $Y := W \times W$ . Set  $\mathbb{Y} := \mathbb{W} \oplus \mathbb{W}$ ,  $Z := W \times \Theta$ , and  $H := \{[u, v] \in Y : u \perp v\}$ . Then there is a nondegenerate form  $\zeta$  on  $Y$  such that  $\text{Sub}_1(H) = \mathcal{Q}_1(\zeta)$ .

Note that  $Z$  is a maximal isotropic subspace of  $(Y, \perp)$  and thus the geometry  $\mathfrak{Q} := \mathcal{Q}_\zeta(\mathbb{Y})$  is a, so called, hyperbolic polar space (or a hyperbolic quadric following [15, Sec. 1.3.4, p. 30], cf. also [4]).

Now let  $\mathcal{Z}$  be a maximal (i.e. a  $(n-1)$ -dimensional) singular subspace of a hyperbolic polar space  $\mathfrak{Q}$  of index  $n-1$  and let  $\mathfrak{R} = \mathfrak{R}(\mathfrak{Q}, \mathcal{Z})$  be the structure obtained by deleting the subspace  $\mathcal{Z}$  from  $\mathfrak{Q}$ . In particular we write  $\mathfrak{R}(\mathbb{W}, \xi) = \mathfrak{R}(\mathcal{Q}_\zeta(\mathbb{Y}), \text{Sub}_1(Z))$ .

**THEOREM 1.1.** *The hyperbolic polar space  $\mathfrak{Q}$  is definable in its reduct  $\mathfrak{R}$ .*

PROOF. We need to recover from  $\mathfrak{R}$  the points and lines of  $\mathcal{Z}$  which are missing to get  $\mathfrak{Q}$ . Let  $\mathcal{C}$  be the family of the maximal singular subspaces of  $\mathfrak{Q}$  and  $\mathcal{R}$  be the family of maximal singular subspaces of  $\mathfrak{R}$ . It is seen that  $\mathcal{R} = \{\mathcal{X} \setminus \mathcal{Z} : \mathcal{X} \in \mathcal{C}\}$ , and thus each element of  $\mathcal{R}$  carries the geometry of a slit space (cf. [11], [12]). Write  $\mathcal{R}_1 = \{\mathcal{X} \setminus \mathcal{Z} : \dim(\mathcal{X} \cap \mathcal{Z}) = n-2\}$  and  $\mathcal{R}_0 = \{\mathcal{X} \setminus \mathcal{Z} : \dim(\mathcal{X} \cap \mathcal{Z}) = 0\}$ . So,

- $\mathcal{R}_1$  consists of the elements of  $\mathcal{R}$  which carry the affine geometry. Each  $\mathcal{J} \in \mathcal{R}_1$  determines on  $\mathcal{Z}$  a  $(n-2)$ -subspace of its “improper” points.
- $\mathcal{R}_0$  consists of the elements of  $\mathcal{R}$  which carry the geometry of a punctured projective space. Each  $\mathcal{J} \in \mathcal{R}_0$  determines on  $\mathcal{Z}$  a point: its “improper” point.

Next, we need some theory of hyperbolic polar spaces (cf. [15, Sec. 1.3.6, p. 35]). In the class  $\mathcal{C}$  we define the relation:  $\mathcal{X}_1 \approx \mathcal{X}_2$  iff  $2 | (\dim(\mathcal{X}_1) - \dim(\mathcal{X}_1 \cap \mathcal{X}_2)) = n-1 - \dim(\mathcal{X}_1 \cap \mathcal{X}_2)$ ; it is an equivalence relation. Take  $\mathcal{J}_0 \in \mathcal{R}_0$ ,  $\mathcal{J}_0 = \mathcal{X}_0 \setminus \mathcal{Z}$ , and  $\mathcal{J}_1 \in \mathcal{R}_1$ ,  $\mathcal{J}_1 = \mathcal{X}_1 \setminus \mathcal{Z}$  where  $\mathcal{X}_0, \mathcal{X}_1 \in \mathcal{C}$ . Set  $J_0 = \mathcal{X}_0 \cap \mathcal{Z}$ ,  $J_1 = \mathcal{X}_1 \cap \mathcal{Z}$ . Then

$$J_0 \mid J_1 \quad \text{iff} \quad \text{there exists a line } L \text{ of } \mathfrak{R} \text{ such that } L \subset J_0 \cap J_1. \quad (1)$$

We write  $\mathcal{J}_0 \mid \mathcal{J}_1$  when the right-hand of (1) holds. Assume first that  $J_0 \mid J_1$ . Then  $J_0 \mid \mathcal{X}_0, \mathcal{X}_1$ . From assumptions  $\mathcal{Z} \not\approx \mathcal{X}_1$ . In case  $2 \mid n-1$  we have  $\mathcal{Z} \approx \mathcal{X}_0$ , so  $\mathcal{X}_0 \not\approx \mathcal{X}_1$ . Hence  $\dim(\mathcal{X}_0 \cap \mathcal{X}_1) > 0$  which means that  $\mathcal{X}_0, \mathcal{X}_1$  share a line  $L'$ . Clearly,  $L' \not\subset \mathcal{Z}$ , so  $L = L' \setminus \mathcal{Z}$  is a required line of  $\mathfrak{R}$ . In case  $2 \nmid n-1$  we get that  $\mathcal{Z} \not\approx \mathcal{X}_0$  and thus  $\mathcal{X}_0 \approx \mathcal{X}_1$  and the above reasoning can be applied again. Conversely, assume that  $\mathcal{J}_0, \mathcal{J}_1$  share a line  $L$ ; then  $L = L' \setminus \mathcal{Z}$  for a line of  $\mathfrak{Q}$  contained in  $\mathcal{X}_0, \mathcal{X}_1$ . Take  $p = L' \cap \mathcal{Z}$ : the suitable improper point. So  $p \mid J_1$ . On the other hand  $p$  is the unique improper point of  $\mathcal{J}_0$ , i.e.  $p = J_0$ . Now, consider the relation  $\simeq$  defined in the class  $\mathcal{R}_0$  as follows

$$\mathcal{J}'_0 \simeq \mathcal{J}''_0 \quad \text{iff} \quad \text{for all } \mathcal{J}_1 \in \mathcal{R}_1 \text{ we have } (\mathcal{J}'_0 \mid \mathcal{J}_1 \iff \mathcal{J}''_0 \mid \mathcal{J}_1). \quad (2)$$

Note that this is an equivalence relation and its equivalence classes can be identified with points on  $\mathcal{Z}$ . In turn, as  $\mathfrak{Q}$  is of type D (i.e. it is hyperbolic), the elements

of  $\mathcal{R}_1$  can be identified with the hyperplanes of  $\mathcal{Z}$  quite naturally. That way, in terms of  $\mathfrak{R}$ , we get an incidence structure with points and hyperplanes of  $\mathcal{Z}$ . Using standard methods we are able to recover lines of  $\mathcal{Z}$  from this incidence structure which makes the proof complete.  $\square$

### 1.3 Symplectic affine polar spaces

From now on  $\xi$  is a nondegenerate symplectic form of index  $m$ . Then  $n = \dim(\mathbb{W}) = 2m$ . Assume that  $m \geq 2$ . The polar space

$$\mathfrak{Q} := \mathfrak{Q}_\xi(\mathbb{W}) = \langle \mathfrak{Q}_1(\xi), \mathfrak{Q}_2(\xi), \subset \rangle$$

is frequently referred to as a *null system* (cf. [3], [9, Vol. 2, Ch. 9, Sec. 3]). Since  $\xi$  is symplectic,  $\mathfrak{Q}_1(\xi) = \text{Sub}_1(\mathbb{W})$  so, the point sets of  $\mathfrak{Q}$  and of  $\mathfrak{P}$  coincide.

Let  $\mathcal{H}_0$  be a hyperplane of  $\mathfrak{Q}$  (cf. [5]); then  $\mathcal{H}_0$  is determined by a hyperplane  $\mathcal{H}$  of  $\mathfrak{P}$ ; on the other hand  $\mathcal{H}$  is a polar hyperplane of a point  $U$  of  $\mathfrak{P}$  i.e.  $\mathcal{H} = U^\perp$ . Finally,  $\mathcal{H}_0 = \mathcal{H}$  is the set of all the points that are collinear in  $\mathfrak{Q}$  with the point  $U$  of  $\mathfrak{Q}$ . The *affine polar space*  $\mathfrak{A}$  derived from  $(\mathfrak{Q}, U)$  is the restriction of  $\mathfrak{Q}$  to the complement of  $\mathcal{H}$ ; in view of the above the point set of  $\mathfrak{A}$  is the point set of the affine space  $\mathfrak{A}$  obtained from  $\mathfrak{P}$  by deleting its hyperplane  $\mathcal{H}$ . The set  $\mathcal{G}$  of lines of  $\mathfrak{A}$  is a subclass of the set  $\mathcal{L}$  of the lines of  $\mathfrak{A}$ . Moreover, the parallelism of the lines in  $\mathcal{G}$  defined as in [5] (two lines are parallel iff they intersect in  $\mathcal{H}_0$ ) coincides with the parallelism of  $\mathfrak{A}$  restricted to  $\mathcal{G}$ . Clearly, not every line of  $\mathfrak{P}$  that is not contained in  $\mathcal{H}$  and which crosses  $\mathcal{H}$  in a point  $U'$  is isotropic. Moreover, none of the lines of  $\mathfrak{P}$  through  $U$  which is not contained in  $\mathcal{H}$  is isotropic. For this reason, in every direction of  $\mathfrak{A}$ , except the one determined by  $U$ , there is a pair of parallel lines in  $\mathfrak{A}$  such that one of them is isotropic and the other is not. In this exceptional direction no line is isotropic.

In [16] affine polar spaces determined in metric affine spaces associated with symmetric forms were studied. Slightly similar interpretation of  $\mathfrak{A}$  can be given here as well.

Recall that there is a basis of  $\mathbb{W}$  in which the form  $\xi$  is given by the formula

$$\xi(x, y) = (x_1 y_2 - x_2 y_1) + (x_3 y_4 - x_4 y_3) + \dots = \sum_{i=1}^m (x_{2i-1} y_{2i} - x_{2i} y_{2i-1}).$$

We write  $\langle u, v, \dots \rangle$  for the vector subspace spanned by  $u, v, \dots$  and  $[x, y, z, \dots]$  for the vector with coordinates  $x, y, z, \dots$  (in some cases  $x, y, \dots$  may be vectors too).

Let us take  $U = \langle [0, 1, 0, \dots, 0] \rangle$ ; then  $\mathcal{H}$  is characterized by the condition  $\langle [x_1, \dots, x_n] \rangle \subset \mathcal{H}$  iff  $x_1 = 0$ . We write  $\mathbb{V}$  for the subspace of  $\mathbb{W}$  characterized by  $x_1 = x_2 = 0$ ; note that the restriction  $\eta$  of  $\xi$  to  $\mathbb{V}$  is also a nondegenerate symplectic form. We can write  $\mathbb{W} = \mathfrak{F} \oplus \mathfrak{F} \oplus \mathbb{V}$  and then for scalars  $a_1, a_2, b_1, b_2$  and vectors  $u_1, u_2$  of  $\mathbb{V}$  we have

$$\xi([a_1, b_1, u_1], [a_2, b_2, u_2]) = a_1 b_2 - a_2 b_1 + \eta(u_1, u_2). \quad (3)$$

Moreover,  $\mathfrak{A} = \mathbf{A}(\mathbb{Y})$  where  $\mathbb{Y} = \mathfrak{F} \oplus \mathbb{V}$ . A point  $[a, u]$  ( $a$  in  $\mathfrak{F}$ ,  $u \in \mathbb{V}$ ) of  $\mathfrak{A}$  can be identified with the subspace  $\langle [1, a, u] \rangle$  of  $\mathbb{W}$ , and the (affine) direction of the line  $[a, u] + \langle [b, w] \rangle$  is identified with the (projective) point  $\langle [0, b, w] \rangle$ .

**LEMMA 1.2.** *Let  $L = [a, u] + \langle [b, w] \rangle$  with  $a, b \in \mathfrak{F}$ ,  $u, w \in \mathbb{V}$  be a line of  $\mathfrak{A}$ . Then*

$$L \in \mathcal{G} \iff \eta(u, w) = -b. \quad (4)$$

PROOF. The projective completion  $\overline{L}$  of  $L$  in  $\mathfrak{P}$  is determined by the pair  $U_1 = \langle [1, a, u] \rangle$ ,  $U_2 = \langle [0, b, w] \rangle$  of points of  $\mathfrak{P}$ . Then  $L \in \mathcal{G}$  iff  $\xi(U_1, U_2) = 0$ . With (3) we get the claim.  $\square$

For  $[a_1, u_1], [a_2, u_2] \in \mathbb{Y}$  we define

$$\rho([a_1, u_1], [a_2, u_2]) := \eta(u_1, u_2) - (a_1 - a_2). \quad (5)$$

**LEMMA 1.3.** *Let  $p_1 = [a_1, u_1]$ ,  $p_2 = [a_2, u_2]$  with  $a_1, a_2 \in \mathfrak{F}$ ,  $u_1, u_2 \in \mathbb{V}$  be a pair of points of  $\mathfrak{A}$ . Then*

$$p_1, p_2 \text{ are collinear in } \mathfrak{A} \iff \eta(u_1, u_2) = a_1 - a_2 \iff \rho(p_1, p_2) = 0. \quad (6)$$

PROOF. As in 1.2, we embed given points into  $\mathfrak{P}$ ; then  $p_i$  corresponds to  $U_i = \langle [1, a_i, u_i] \rangle$ . Since  $p_1, p_2$  are collinear iff the projective line which joins  $U_1, U_2$  is in  $\mathfrak{Q}$  we get that  $p_1, p_2$  are collinear iff  $\xi(U_1, U_2) = 0$ . With (3) we have the claim.  $\square$

In what follows we write  $a \sim b$  if points  $a, b$  of  $\mathfrak{A}$  are collinear in  $\mathfrak{A}$ .

From 1.2 and 1.3 we learn that the affine polar space  $\mathfrak{A}$  can be defined entirely in terms of a vector space  $\mathbb{V}$  over  $\mathfrak{F}$  and a nondegenerate symplectic form  $\eta$  on  $\mathbb{V}$ .

Note that the surrounding affine space  $\mathfrak{A}$  is definable in terms of the geometry of  $\mathfrak{A}$ . The result is a simple consequence of elementary properties of (symplectic) polar spaces, but it is important from the view of “foundations”: *the geometry of symplectic aps'es can be expressed in the language of the relation  $\sim$ .*

**THEOREM 1.4.** *Let  $p, q$  be two distinct points of  $\mathfrak{A}$ . Then*

$$\bigcap \{ \{x : x \sim y\} : y \sim p, q \}$$

*is the line of  $\mathfrak{A}$  that passes through  $p, q$ . Consequently, the structure  $\mathfrak{A}$  is definable in terms of the binary collinearity of  $\mathfrak{A}$  and thus it is definable in  $\mathfrak{A}$  as well.*

PROOF. It suffices to note that  $H_y = \{x : x \sim y\}$  is a polar hyperplane of a point  $y$  and if  $y \sim p, q$ , then  $p, q \in H_y$ .  $\square$

Interpretation of the isotropic (singular) subspaces of higher dimensions in  $\mathfrak{A}$  that makes use of the map  $\rho$ , analogous to 1.2, remains, clearly, valid. An interested reader can consider this problem as an easy exercise.

## 2 Semiforms

The construction of the function  $\rho$  in (5) falls into the following more general one.

**DEFINITION 2.1.** Let  $\mathbb{V}, \mathbb{V}'$  be vector spaces over a (commutative) field  $\mathfrak{F}$  with  $\text{char}(\mathfrak{F}) \neq 2$ . Let  $V, V'$  be their sets of vectors and  $\theta, \mathbf{0}$  be their zero-vectors, respectively.

(i) Let  $\eta : V \times V \longrightarrow V'$  be an alternating bilinear map. Then  $\eta(u_1, u_2) = -\eta(u_2, u_1)$  and  $\eta(u, u) = \mathbf{0}$  for all vectors  $u, u_1, u_2 \in V$ .

(ii) Let  $\delta : V' \times V' \longrightarrow V'$  be a map that satisfies the following conditions

$$\text{C1. } \delta(v_1 + v, v_2 + v) = \delta(v_1, v_2),$$

- C2.  $\delta(\alpha v_1, \alpha v_2) = \alpha \delta(v_1, v_2),$   
 C3.  $\delta(v_1, v) + \delta(v, v_2) = \delta(v_1, v_2).$

for all scalars  $\alpha$  and  $v, v_1, v_2 \in V'$ .

Set  $Y := V' \times V$  and  $\mathbb{Y} := \mathbb{V}' \oplus \mathbb{V}$ . On  $Y$  we define the binary operation  $\varrho: Y \times Y \rightarrow V'$  by the formula

$$\varrho([v_1, u_1], [v_2, u_2]) := \eta(u_1, u_2) - \delta(v_1, v_2). \quad (7)$$

The resulting map  $\varrho$  is referred to as a *semiform* defined on  $\mathbb{Y}$ .

An alternating bilinear form  $\eta$  considered in 2.1 is *nondegenerate* when for each  $\theta \neq u_1 \in V$  there is  $u_2 \in V$  such that  $\eta(u_1, u_2) \neq 0$ .

The following technical but important formulas are immediate from definition. Let  $p_i = [v_i, u_i]$ ,  $q = [v, y]$ .

$$\varrho(\alpha p_1, \alpha p_2) - \alpha \varrho(p_1, p_2) = \alpha(\alpha - 1)\eta(u_1, u_2); \quad (8)$$

$$\varrho(p_1 + q, p_2 + q) - \varrho(p_1, p_2) = \eta(u_1 - u_2, y); \quad (9)$$

in particular,

$$\varrho(p_1, p_1 + p_2) - \varrho(\theta, p_2) = \eta(u_1, u_2). \quad (10)$$

$$\varrho(q, \theta) = v, \quad (11)$$

$$\varrho(\alpha p_1, q) - \alpha \varrho(p_1, q) = (1 - \alpha)v, \quad (12)$$

$$\varrho(p_1 + p_2, q) - (\varrho(p_1, q) + \varrho(p_2, q)) = -v. \quad (13)$$

One example is crucial:

**EXAMPLE 2.2.** Let  $\eta$  be a null-form defined on  $\mathbb{V}$ . Next, let  $\mathbb{V}' = \mathfrak{F}$  and  $\delta(a, b) = a - b$ . Definition 2.1 coincides with the definition of the function  $\rho$  in (5).

**EXAMPLE 2.3.** Each alternating map  $\eta: V \times V \rightarrow V'$  is derived from a linear map  $g: \bigwedge^2 \mathbb{V} \rightarrow V'$  by the formula

$$\eta(u_1, u_2) = g(u_1 \wedge u_2). \quad (14)$$

(see any standard textbook, e.g. [13, Ch. XIX]).

We shall write, generally, (cf. (16))  $\eta_u$  for the map defined by  $\eta_u(v) = \eta(u, v)$ . It is a folklore that  $\dim(\bigwedge^2 \mathbb{V}) = \binom{n}{2}$ , where  $n = \dim(\mathbb{V})$ . Note that when  $u$  is fixed then the set  $S_u := \{u \wedge y: y \in V\} = \text{Im}(\wedge_u)$  is a  $(n-1)$ -dimensional vector subspace of  $\bigwedge^2 \mathbb{V}$ .

Clearly, the operation  $\eta = \wedge$  together with a given  $\delta$  determines via (7) a semiform.

**EXAMPLE 2.4.** Let  $\mathbb{V}$  be a 3-dimensional vector space. Then  $\bigwedge^2 \mathbb{V} \cong \mathbb{V}$  and we can write  $u' \wedge u'' = u' \times u''$ , where  $\times: V \times V \rightarrow V$  is a vector product defined on  $\mathbb{V}$ . A standard formula defining  $\times$  is the following:

$$[\alpha'_1, \alpha'_2, \alpha'_3] \times [\alpha''_1, \alpha''_2, \alpha''_3] = \left[ \varepsilon_1 \begin{vmatrix} \alpha'_2 & \alpha'_3 \\ \alpha''_2 & \alpha''_3 \end{vmatrix}, \varepsilon_2 \begin{vmatrix} \alpha'_1 & \alpha'_3 \\ \alpha''_1 & \alpha''_3 \end{vmatrix}, \varepsilon_3 \begin{vmatrix} \alpha'_1 & \alpha'_2 \\ \alpha''_1 & \alpha''_2 \end{vmatrix} \right]$$

with  $\varepsilon_i = \pm 1$  (cf. [17], [10]). Then  $\varrho$  defined on  $\mathbb{V} \oplus \mathbb{V}$  by the formula

$$\varrho([v_1, u_1], [v_2, u_2]) = u_1 \times u_2 - (v_1 - v_2)$$

is a semiform.

## 2.1 Affine atlas and its characterization

In this and the forthcoming subsections 2.2 and 2.3 most of the proofs consist in direct computations and therefore they are left for the reader.

Let us give a more explicit representation of a map  $\delta$  characterized in 2.1(ii).

**LEMMA 2.5.** *Let  $\delta$  meet conditions C1-C3 of 2.1(ii). Then the following conditions follow as well:*

- C4.  $\delta(\mathbf{0}, \mathbf{0}) = \mathbf{0}$  (by C2);
- C5.  $\delta(v, v) = \mathbf{0}$  (by C1, C4);
- C6.  $\delta(v_1, v_2) = -\delta(v_2, v_1)$  (by C3, C5);
- C7.  $\delta(v_1 + v_2, \mathbf{0}) = \delta(v_1, \mathbf{0}) + \delta(v_2, \mathbf{0})$  (by C1 - C3, C6);

for all  $v, v_1, v_2 \in V'$ .

Define  $\phi: V' \rightarrow V'$  by the formula  $\phi(v) = \delta(v, \mathbf{0})$ . Then  $\phi$  is a linear map and  $\delta$  is characterized by the formula

$$\delta(v_1, v_2) = \phi(v_1) - \phi(v_2) \quad (= \phi(v_1 - v_2)). \quad (15)$$

A map  $\delta$  defined by formula (15) is called *an affine atlas*, it is *nondegenerate* when  $\phi$  is an injection (i.e. if  $\ker(\phi)$  is trivial). Note that when  $\dim(V') < \infty$  and  $\delta$  is nondegenerate then the representing map  $\phi$  is a surjection as well.

The following is straightforward

**LEMMA 2.6.** *Let  $\phi: V' \rightarrow V'$  be a linear map and  $\delta$  be defined by (15). Then  $\delta$  meets conditions C1-C3 of 2.1(ii).*

Finally, we note that affine atlases can be equivalently characterized by another, less elegant but more convenient for our further characterizations, set of postulates.

**LEMMA 2.7.** *Let  $\delta$  satisfy the postulates C2, C7, C6, C1 of 2.1(ii), 2.5. Then  $\delta$  satisfies C3 as well.*

## 2.2 Synthetic characterization and representations of semiforms

Let  $\mathbb{Y} = \langle Y, +, \theta \rangle$ ,  $\mathbb{Z} = \langle Z, +, \mathbf{0} \rangle$  be vector spaces with the common field  $\mathfrak{F}$  of scalars. Let  $\varrho: Y \times Y \rightarrow Z$  be a map. Consider the following properties:

- A1.  $\varrho(p, q) = -\varrho(q, p)$  for each  $p, q \in Y$ .
- A2. If  $\varrho(\theta, p) = \mathbf{0}$  then  $\varrho(\alpha q, p) = \alpha \varrho(q, p)$  for each scalar  $\alpha$  and each vector  $q$ .
- A3. If  $\varrho(\theta, p) = \mathbf{0}$  then  $\varrho(q_1 + q_2, p) = \varrho(q_1, p) + \varrho(q_2, p)$ .
- A4. If  $p \neq \theta$  then there is  $q$  with  $\varrho(p, q) \neq \mathbf{0}$  and  $\varrho(\theta, q) = \mathbf{0}$ .
- A5.  $\varrho(-p, -q) + \varrho(p, q) = 2(\varrho(p, p + q) - \varrho(\theta, q))$ .
- A6.  $(\forall p)[\varrho(p + q, q) = \varrho(p, \theta)]$  implies  $(\forall p_1, p_2)[\varrho(p_1 + q, p_2 + q) = \varrho(p_1, p_2)]$ .
- A7.  $2(\varrho(\alpha p_1, \alpha p_2) - \alpha \varrho(p_1, p_2)) = \alpha(\alpha - 1)(\varrho(-p_1, -p_2) + \varrho(p_1, p_2))$ .
- A8. For each  $q \in Y$  there is  $p \in Y$  such that  $\varrho(p, \theta) = \mathbf{0}$  and  $\varrho(p - q, -r) = -\varrho(q - p, r)$  for all  $r \in Y$ .



In view of formulas (8) – (13) it is evident that Axioms A1 – A8 are satisfied by each semiform as defined in (7).

Set  $M := \{p \in Y : \varrho(\theta, p) = \mathbf{0}\}$ . With each  $p \in Y$  we associate the map

$$\varrho_p : Y \longrightarrow Z, \quad \varrho_p(q) = \varrho(q, p). \quad (16)$$

Note that if  $\varrho$  is a semiform defined in 2.1 then  $M = V$  and  $\varrho \upharpoonright M \times M = \eta$ .

Recall that in one of the most intensively investigated cases in geometry when we consider a sesquilinear form  $\varrho$ ,  $M = Y$ ,  $\varrho_p$  is a linear map, and  $p \mapsto \varrho_p$  is semilinear. Our axioms lead to a similar situation.

**LEMMA 2.8.** *If  $\varrho$  satisfies A1 then  $\varrho(p, p) = \mathbf{0}$  for each  $p \in Y$ . Consequently,  $\theta \in M$ .*

**LEMMA 2.9.** *Assume Axiom A1. Let  $p \in Y$ .*

- (i) *If  $\varrho_p$  is additive then  $p \in M$ .*
- (ii) *If  $\varrho_p$  is multiplicative then  $p \in M$ .*

*Consequently, if Axioms A2 and A3 are valid then the map  $\varrho_p$  is linear iff  $p \in M$ .*

*In particular (cf. 2.8),  $\varrho_\theta$  is a linear map, i.e. the following hold:*

$$\begin{aligned} \varrho(\alpha p, \theta) &= \alpha \varrho(p, \theta), \\ \varrho(p_1 + p_2, \theta) &= \varrho(p_1, \theta) + \varrho(p_2, \theta). \end{aligned}$$

*Clearly,  $M = \ker(\varrho_\theta)$  and thus  $M$  is a subspace of  $\mathbb{Y}$ .*

*If, moreover, Axiom A4 is valid then the assignment  $M \ni p \mapsto \varrho_p$  is injective.*

**LEMMA 2.10.** (i) *Set  $D' := \{q \in Y : (\forall p \in Y)[\varrho(q, q + p) = \varrho(\theta, p)]\}$ .*

*Then  $\theta \in D'$  and the set  $D'$  is closed under vector addition.*

(ii) *Assume Axiom A6. Then  $q \in D'$  iff the condition  $\varrho(p_1 + q, p_2 + q) = \varrho(p_1, p_2)$  holds for all  $p_1, p_2 \in Y$ .*

(iii) *Set  $D'' := \{q \in Y : (\forall p \in Y)[\varrho(-p, -q) = -\varrho(p, q)]\}$ .*

*If Axiom A7 is valid then the set  $D''$  is closed under scalar multiplication.*

*If Axiom A5 is adopted then  $D' = D''$ .*

*Consequently, if Axioms A6, A7, and A5 are valid then  $D := D' = D''$  is a vector subspace of  $\mathbb{Y}$ .*

*Moreover, if Axioms A1-A7 are valid then  $M \cap D = \{\theta\}$ .*

**LEMMA 2.11.** *With the Axioms A1-A7, Axiom A8 can be expressed as the following statement:*

$$\mathbb{Y} = D \oplus M.$$

*Assume that the Axioms A1-A8 are valid and set  $\eta := \varrho \upharpoonright M \times M$ ,  $\delta := \varrho \upharpoonright D \times D$ . Then  $\eta$  is an alternating nondegenerate vector-valued form. The map  $\delta$  is a non-degenerate affine atlas; it is determined by a linear injection  $\phi : D \longrightarrow D$  by the formula (15).*

**LEMMA 2.12.** *Let  $q_i = p_i + r_i$  with  $p_i \in M$ ,  $r_i \in D$  for  $i = 1, 2$ . Then, we have (cf. (7)) the following*

$$\varrho(q_1, q_2) = \eta(p_1, p_2) - \delta(r_1, r_2). \quad (17)$$

Summing up the above, with not too tedious computation, we close this part by the following representation theorem

**THEOREM 2.13.** *Let  $\varrho: Y \times Y \longrightarrow Z$  be a map. The following conditions are equivalent.*

- (i)  $\varrho$  is a semiform defined in accordance with 2.1, where  $\eta, \delta$  are nondegenerate.
- (ii)  $\varrho$  satisfies Axioms A1-A8.

**Remark.** A nondegenerate semiform  $\varrho$  is scalar valued (i.e.  $\dim(Z) = 1$ ) iff it is associated with a symplectic polar space.

**EXAMPLE.** Let  $\dim(\mathbb{V}) = 2$ . Then the determinant is a symplectic form. Therefore the map below ( $x_i, y_i$  are elements of the field of scalars  $\mathfrak{F}$  of  $\mathbb{V}$ )

$$\varrho([x_1, x_2, x_3], [y_1, y_2, y_3]) = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - (x_1 - y_1)$$

is a semiform. The associated aps is determined by the so called line complex in the 3-dimensional projective space over  $\mathfrak{F}$  (cf. [7, Ch. 6], [9, Vol. 2, Ch. 9, Sec. 3]).

### 2.3 A simplification of semiforms

Forthcoming constructions are provided for a fixed nondegenerate semiform  $\varrho$  defined in 2.1. Moreover, we assume that

$$\dim(\mathbb{V}') =: \nu < \infty.$$

Set  $\mathfrak{A} = \mathbf{A}(\mathbb{Y})$ . Let  $p_1, p_2$  be vectors of  $\mathbb{Y}$ , so  $p_i = [v_i, u_i]$ ,  $v_i \in V'$ ,  $u_i \in V$ . By definition,

$$\varrho(p_1, p_2) = \eta(u_1, u_2) - \phi(v_1 - v_2)$$

for suitable maps  $\eta, \phi$  (recall, they need to be nondegenerate. As a consequence,  $\phi \in GL(\mathbb{V}')$ ).

There is, generally, a great variety of semiforms. But some of them may lead to isomorphic geometries. Write  $\varrho_{\eta, \phi}$  for  $\varrho$  defined by 2.1 with  $\delta$  defined by (15). We have evident

**PROPOSITION 2.14.**

- (i) *There is a linear bijection  $\Phi \in GL(\mathbb{Y})$  such that for any  $q_1, q_2 \in Y$  it holds:*

$$\varrho_{\eta, \phi}(q_1, q_2) = \varrho_{\eta, \text{id}}(\Phi(q_1), \Phi(q_2))$$

- (ii) *Let  $B \in GL(\mathbb{V})$ ,  $\gamma$  be a non zero scalar. Then, clearly, the map  $\gamma\eta B$  defined by  $\gamma\eta B(u_1, u_2) = \gamma \cdot \eta(B(u_1), B(u_2))$  is an alternating form. There is a linear bijection  $\Phi \in GL(\mathbb{V})$  such that the following holds for any  $q_1, q_2 \in Y$*

$$\varrho_{\gamma\eta B, \text{id}}(q_1, q_2) = \gamma^{-1} \cdot \varrho_{\eta, \text{id}}(\Phi(q_1), \Phi(q_2)).$$

**Remark.** In terms of 2.3 we have  $\eta B = g \circ (B \wedge B)$ .

In view of 2.14, till the end of our paper we assume that  $\varrho$  is defined by the formula of the form

$$\varrho([v_1, u_1], [v_2, u_2]) = \eta(u_1, u_2) - (v_1 - v_2). \quad (18)$$

## 2.4 Affine semipolar spaces

Imitating 1.3, for points  $p_1, p_2$  of  $\mathfrak{A}$ , we put generally

$$p_1 \sim p_2 \iff \varrho(p_1, p_2) = \mathbf{0}. \quad (19)$$

From definition it is immediate that  $p_1 \sim p_2 \iff \eta(u_1, u_2) = v_1 - v_2$ .

**LEMMA 2.15.** *Let  $p_1, p_2$  be two distinct points of  $\mathfrak{A}$  and  $L = \overline{p_1, p_2}$ . If  $p_1 \sim p_2$  then  $q_1 \sim q_2$  for all  $q_1, q_2 \in L$ .*

PROOF. Let us write  $p_1 = [v, u]$ ,  $p_2 = p_1 + q$ , and  $q = [v_0, u_0]$ . From assumption,  $\eta(u, u+u_0) = v - (v+v_0) = -v_0$ . We directly compute that then  $\eta(u+\alpha u_0, u+\beta u_0) = (\alpha - \beta)\eta(u, u_0) = (\alpha - \beta)v_0 = (v + \alpha v_0) - (v + \beta v_0)$ . This yields  $p_1 + \alpha q \sim p_1 + \beta q$  for any scalars  $\alpha, \beta$  and closes the proof.  $\square$

In view of 2.15, the relation  $\sim$  determines the class  $\mathcal{G}$  of lines of  $\mathfrak{A}$  by the condition

$$\begin{aligned} L \in \mathcal{G} \text{ iff } p_1 \sim p_2 \text{ for any } p_1, p_2 \in L; \\ \text{equivalently, iff } p_1 \sim p_2 \text{ for a pair } p_1, p_2 \text{ of distinct points on } L. \end{aligned} \quad (20)$$

For computation it is convenient to have this criteria (comp. 1.2):

$$[v_0, u_0] + \langle [v, u] \rangle \in \mathcal{G} \text{ iff } \eta(u_0, u) = -v. \quad (21)$$

It is a straightforward consequence of (20) and (19).

The class  $\mathcal{G}$  induces the incidence structure  $\langle Y, \mathcal{G} \rangle$  that we will take a look into. Let us call this structure the *affine semipolar space* determined by  $\varrho$ .

**LEMMA 2.16.**

(i) *The class  $\mathcal{G}$  is unclosed under parallelism, i.e. for every  $L_1 \in \mathcal{G}$  there is an affine line  $L_2 \notin \mathcal{G}$  such that  $L_1 \parallel L_2$ .*

(ii) *Let  $L_1, L_2 \in \mathcal{G}$ ,  $L_1 \neq L_2$ , and  $p \in L_1 \cap L_2$ . If  $L$  is an affine line through  $p$  from the affine plane  $\langle L_1, L_2 \rangle$ , then  $L \in \mathcal{G}$ .*

PROOF. (i): Straightforward computation.

(ii): Without loss of generality we can assume that  $p = [v_0, u_0]$  and  $L_i = p + \langle a_i \rangle$  where  $a_i \in Y$ ,  $i = 1, 2$ . Then  $L = p + \langle \alpha_1 a_1 + \alpha_2 a_2 \rangle$  for some  $\alpha_i \in F$ . Applying (21) to  $L_1, L_2$  and then to  $L$  we are through.  $\square$

**THEOREM 2.17.** *The affine semipolar space determined by a semiform is a  $\Gamma$ -space and its every singular subspace carries affine geometry.*

PROOF. Let  $\langle Y, \mathcal{G} \rangle$  be our affine semipolar space. The first part follows directly from 2.16(ii). The other part is a simple observation that a singular subspace of  $\langle Y, \mathcal{G} \rangle$ , in other words, a strong subspace wrt.  $\sim$  in  $\mathfrak{A}$ , is an affine subspace of  $\mathfrak{A}$ .  $\square$

When we deal with a  $\Gamma$ -space a question on the form of its triangles may appear important. The following is immediate from (21) and (19).

**Remark 1.** A triangle in an affine semipolar space  $\langle Y, \mathcal{G} \rangle$  has form

$$[v_0, u_0], \quad [v_0 + \eta(u, u_0), u_0 + u], \quad [v_0 + \eta(y, u_0), u_0 + y], \quad (22)$$

where  $\eta(u, y) = \mathbf{0}$ .

**COROLLARY 2.18.** *If  $\dim(\ker(\eta_u)) = 1$  for each nonzero vector  $u \in V$  then the corresponding affine semipolar space contains no proper triangle. In that case its maximal singular subspaces are the lines.*

**EXAMPLE 2.19.** In view of 2.17 one could expect that affine semipolar spaces are models of the system considered in [6].

In case considered in 2.3 and, consequently, in case considered in 2.4 we have  $\dim(\ker(\eta_u)) = 1$  for all  $u \neq \theta$  and corresponding alternating map  $\eta$ . Therefore, *the structure  $\langle Y, \mathcal{G} \rangle$  has no triangles. So, affine semipolar spaces determined by them are not models of the system considered in [6].*

One can also compute that, e.g. an affine semipolar space determined by the exterior power operation *is not a generalized quadrangle.*

In the sequel we shall frequently consider the condition (with prescribed values  $u, v$ )

$$(\exists y) [\eta(u, y) = v];$$

applying the representation given in 2.3 this can be read as  $(\exists y) [g(u \wedge y) = v]$ , which is equivalent to  $(\exists \omega \in S_u) [g(\omega) = v]$ . This observation allows us to construct quite “strange” (‘locally surjective’) alternating maps.

As an immediate consequence of 2.15 and the definition we have

**LEMMA 2.20.** *Let  $q = [v_0, u_0]$  be a vector of  $\mathbb{Y}$ . The following conditions are equivalent.*

- (i) *There is no line  $L \in \mathcal{G}$  with the direction  $q$ .*
- (ii) *The equation*

$$\eta(u_0, u) = v_0 \quad (23)$$

*is not solvable in  $u$ .*

*In particular, if  $u_0 = \theta$  and  $v_0 \neq \mathbf{0}$  then (23) is not solvable and thus there is no line  $L \in \mathcal{G}$  with the direction  $q$ .*

Set

$$D := \{q \in Y : \text{no line in } \mathcal{G} \text{ has the direction } q\}$$

Note, in particular, that when  $\eta_u : V \longrightarrow V'$  is a surjection for each non zero vector  $u$  then  $D = V' \times \{\theta\}$ .

**EXAMPLE-CONTINUATION 2.4-A** Let  $\times$  be a vector product in a vector 3-space  $\mathbb{V}$  associated with a nondegenerate bilinear symmetric form  $\xi$  and  $\perp = \perp_\xi$  be the orthogonality determined by  $\xi$ . Then for  $u_0, v_0 \neq \theta$  equation (23) is solvable iff  $u_0 \perp v_0$ . In that case we have

$$D = V \times \{\theta\} \cup \{[v, u] \in V \times V : u \not\perp v\}.$$

**LEMMA 2.21.** *For a fixed  $u_0 \in V$ ,  $v_0 \in V'$  and a scalar  $\alpha$  the set*

$$\mathcal{Z} = \{[v, u] : \eta(u_0, u) = v_0 + \alpha v\} \quad (24)$$

*is a subspace of  $\mathfrak{A}$ . The class of sets of form (24) is invariant under translations of  $\mathfrak{A}$ .*

PROOF. Take  $[v_1, u_1], [v_2, u_2] \in \mathcal{Z}$  and an arbitrary scalar  $\lambda$ . Then we compute  $\eta(u_0, \lambda u_1 + (1 - \lambda)u_2) = \lambda \eta(u_0, u_1) + (1 - \lambda) \eta(u_0, u_2) = \lambda(v_0 + \alpha v_1) + (1 - \lambda)(v_0 + \alpha v_2) = v_0 + (\lambda v_1 + (1 - \lambda)v_2) + \alpha(\lambda u_1 + (1 - \lambda)u_2)$  which proves that  $[\lambda v_1 + (1 - \lambda)v_2, \lambda u_1 + (1 - \lambda)u_2] \in \mathcal{Z}$  and thus  $\overline{[v_1, u_1], [v_2, u_2]} \subset \mathcal{Z}$ . This proves that  $\mathcal{Z}$  is a subspace of  $\mathfrak{A}$ .

Write  $\mathcal{Z}_{u_0, v_0, \alpha}$  for the set defined by (24). Let  $q = [x, y] \in Y$  be arbitrary. Then  $\tau_q([v, u]) = [v + x, u + y] \in \mathcal{Z}_{u_0, v_0, \alpha} \iff \eta(u_0, u + y) = v_0 + \alpha(v + x) \iff \eta(u_0, u) = (v_0 - \eta(u_0, y) + \alpha x) + \alpha v \iff [v, u] \in \mathcal{Z}_{u_0, v_0 - \eta(u_0, y) + \alpha x, \alpha}$ . Thus

$$\tau_q^{-1}(\mathcal{Z}_{u_0, v_0, \alpha}) = \mathcal{Z}_{u_0, v_0 - \eta(u_0, y) + \alpha x, \alpha}.$$

This closes our proof.  $\square$

**LEMMA 2.22.** *Let  $\mathcal{Z}$  be defined by (24). Then either  $\mathcal{Z}$  is an empty set or it is an affine subspace of  $\mathfrak{A}$  with the dimension  $\nu + \dim(\ker(\eta_u))$ , or with the dimension  $\dim(\mathbb{V})$ .*

PROOF. If  $\mathcal{Z}$  is nonempty then by 2.21 we can assume that  $[0, \theta] \in \mathcal{Z}$ . Then  $\mathcal{Z}$  is characterized by an equation  $\eta(u_0, u) = \alpha_0 v$  with prescribed values of  $u_0$ ,  $\alpha_0$  and it is the kernel of the linear map  $\Psi: Y \rightarrow V'$ ,  $\Psi[v, u] \mapsto \eta(u_0, u) - \alpha_0 v$ . If  $\alpha_0 = 0$  then, clearly,  $\mathcal{Z} = V' \times \ker(\eta_{u_0})$  and thus  $\dim(\mathcal{Z}) = \dim(\ker(\eta_{u_0}) + \dim(\mathbb{V}') = \nu + \dim(\ker(\eta_{u_0}))$ . Assume that  $\alpha_0 \neq 0$ ; then  $\mathcal{Z}$  can be considered as the kernel of the map  $[v, u] \mapsto \eta(\frac{1}{\alpha_0}u_0, u) - v$ . Let  $(d_1, \dots, d_k)$  be a linear basis of  $\text{Im}(\eta_{u_0})$  and  $(e_1, \dots, e_m)$  be a basis of  $\ker(\eta_{u_0})$ . Choose one  $z_i \in V$  with  $\eta_{u_0}(z_i) = d_i$  for each  $i = 1, \dots, k$ ; Then the set  $\{z_1, \dots, z_k\}$  is linearly independent. Moreover, the subspaces  $\langle z_1, \dots, z_k \rangle$  and  $\ker(\eta_{u_0})$  have only the zero vector in common. A basis of  $\mathcal{Z}$  consists of the vectors

$$([d_1, z_1], [d_1, z_1 + e_1], \dots, [d_1, z_1 + e_m], [d_2, z_2], \dots, [d_k, z_k]).$$

Consequently,  $\dim(\mathcal{Z}) = \dim(\ker(\eta_{u_0})) + \dim(\text{Im}(\eta_{u_0})) = \dim(\text{Dom}(\eta_{u_0})) = \dim(\mathbb{V})$ .  $\square$

Applying 2.22 and (19) we get, e.g. a geometrically interesting strengthening of 2.16(ii):

**COROLLARY 2.23.** *The set of points that are joinable in an affine semipolar space (determined by a semiform defined on  $\mathbb{V}$ ) with a given point is a subspace of dimension  $\dim(\mathbb{V})$  in the surrounding affine space  $\mathfrak{A}$ .*

Another corollary of analogous type that will appear important in the sequel is read as follows.

**LEMMA 2.24.** *Assume the following:*

(\*) If  $u' \nparallel u''$  are two vectors of  $\mathbb{V}$  then there is  $y_0$  such that  $\eta(u', y_0) = \mathbf{0}$  and  $\eta(u'', y_0) \neq \mathbf{0}$ .

Let  $L \in \mathcal{G}$  pass through  $p = [\mathbf{0}, \theta]$  and  $p'$  be a point on  $L$ . Then  $L = \bigcap \{[q]_{\sim} : q \approx p, p'\}$ .

Finally, let us make a few comments that enable us to characterize (with the help of 1) the geometry of the lines and the planes through a point in an affine semipolar space.

Each alternating map  $\eta: V \times V \longrightarrow V'$  determines the incidence substructure  $\mathbf{Q}_\eta(\mathbb{V})$  of the projective space  $\mathbf{P}(\mathbb{V})$  with the point set unchanged and with the class  $\mathcal{L}^*$  of projective lines of the form  $\langle u', u'' \rangle$ , where  $u', u'' \in V$  are linearly independent and  $\eta(u', u'') = \mathbf{0}$  as its lines. With a fixed basis of  $V'$  one can write  $\eta$  as the (Cartesian) product of  $\nu$  bilinear alternating forms  $\eta_i: V \times V \longrightarrow F$ :

$$\eta(u', u'') = [\eta_1(u', u''), \dots, \eta_\nu(u', u'')], \quad (25)$$

clearly, the  $\eta_i$  need not be nondegenerate. So, each  $\eta_i$  determines a (possibly degenerate) null system  $\mathbf{Q}_{\eta_i}(\mathbb{V})$  with the lines  $\mathbf{Q}_2(\eta_i)$ . The class  $\mathcal{L}^*$  is simply  $\bigcap_{i=1}^\nu \mathbf{Q}_2(\eta_i)$ .

**PROPOSITION 2.25.** *The geometry of the lines and planes of an affine semipolar space (determined by a semiform  $\varrho$  associated via (18) with an alternating map  $\eta$ ) which pass through the point  $[\mathbf{0}, \theta]$  is isomorphic to  $\mathbf{Q}_\eta(\mathbb{V})$ .*

PROOF. Let  $p = [\mathbf{0}, \theta]$ . In view of (21) the class of lines through  $p$  is the set  $\{\langle [\mathbf{0}, u] \rangle : u \text{ is a nonzero vector}\}$  so, it can be identified with the point set of  $\mathbf{P}(\mathbb{V})$  under the map  $\langle [\mathbf{0}, u] \rangle \mapsto \langle u \rangle$ . From 1 we infer that two lines  $\langle [\mathbf{0}, u'] \rangle, \langle [\mathbf{0}, u''] \rangle$  span a plane in the corresponding affine semipolar space iff  $\eta(u', u'') = \mathbf{0}$ , which closes our reasoning.  $\square$

From the homogeneity of each affine semipolar space (which will be proved later in 2.27) one will get that the geometry of the lines and the planes through arbitrary point of an affine semipolar space is (in the above sense) a generalized null system.

## 2.5 Automorphisms

To establish the automorphism group of the relation  $\sim$  we need some additional assumptions. One of these conditions is read as follows:

(\*\*) The set of directions of  $V' \times \{\theta\}$  can be characterized in terms of the projective geometry of the horizon of  $\mathbf{A}(\mathbb{Y})$  with the set of directions of  $D$  distinguished.

Clearly, in view of 2.20 this condition holds when  $\varrho$  is scalar valued. Let us point out that it is not a unique possibility when this condition is valid.

**EXAMPLE-CONTINUATION 2.4-B** We continue with the notation of 2.4. Let  $f \in \Gamma L(Y)$  preserve the set of directions  $D$ . Then  $f$  preserves the vector subspace  $V \times \{\theta\}$ . Indeed, the geometric structure of the complement of  $D$  carries the geometry of the reduct  $\mathfrak{R}(\mathbb{V}, \xi)$  of a hyperbolic polar space of the form considered in 1.2. Our claim follows from 1.1. Consequently, the condition (\*\*) is valid here.

**PROPOSITION 2.26.** *If  $F$  is given by the formula*

$$F([v, u]) = [\psi_1(v) + \psi_2(u) + v_0, \varphi(u) + u_0] \quad (26)$$

where  $v_0 \in V'$ ,  $u_0 \in V$ ,  $\psi_1: V' \rightarrow V'$ ,  $\varphi: V \rightarrow V$  are linear bijections,  $\psi_2: V \rightarrow V'$ , and the following holds:

- a)  $\psi_2(u) = \eta(\varphi(u), u_0)$  for every vector  $u$  of  $\mathbb{V}$ , and
- b)  $\eta(\varphi(u_1), \varphi(u_2)) = \psi_1\eta(u_1, u_2)$ , for all vectors  $u_1, u_2$  of  $\mathbb{V}$ ,

then  $F$  preserves the relation  $\approx$ . In that case the semiform  $\varrho$  is transformed under the rule

$$\varrho(F(p_1), F(p_2)) = \psi_1(\varrho(p_1, p_2)) \quad (27)$$

for any pair  $p_1, p_2$  of points of  $\mathfrak{A}$ .

Conversely, under additional assumption that  $(**)$  is valid, each linear (affine) automorphism of  $\mathfrak{A}$  is of the form (26).

**NOTE.** If  $\eta$  is 'onto'  $V'$  then given map  $\varphi$ , condition b) uniquely determines  $\psi_1$ . Similarly, for a given map  $\varphi$  and vector  $u_0$ , condition a) uniquely determines  $\psi_2$ .

**PROOF.** Assume that  $F$  is defined by the formula (26) and a), b) hold. Let  $p_i = [v_i, u_i]$ ,  $v_i \in V'$ ,  $u_i \in V$ , for  $i = 1, 2$ . We compute as follows:  $\varrho(F(p_1), F(p_2)) = \eta(\varphi(u_1), \varphi(u_2)) + \eta(\varphi(u_1 - u_2), u_0) - \psi_1((v_1 - v_2) - \psi_2(u_1 - u_2)) = \psi_1\eta(u_1, u_2) + \eta(\varphi(u_1 - u_2), u_0) - \psi_1(v_1 - v_2) - \eta(\varphi(u_1 - u_2), u_0) = \psi_1\eta(u_1, u_2) - \psi_1(v_1 - v_2) = \psi_1(\varrho(p_1, p_2))$ , which proves (27). This yields, in particular, that  $F$  preserves  $\approx$ .

Now assume that  $F$  is an affine automorphism of  $\mathfrak{A}$  preserving  $\approx$  and that  $(**)$  is valid. Then,  $F$  is a composition  $\tau_{[v_0, u_0]} \circ F_0$ , where  $F_0 \in GL(\mathbb{Y})$  and  $[v_0, u_0]$  is a vector of  $\mathbb{Y}$ . The map  $F_0$  can be presented in the form  $F_0([v, u]) = [\psi_1(v) + \psi_2(u), \varphi_1(u) + \varphi_2(v)]$  for suitable linear maps  $(\varphi_2: V' \rightarrow V)$ . By 2.20, the linear part  $F_0$  of  $F$  fixes the subspace  $V'$  and thus  $\varphi_2 \equiv \theta$ . We write  $\varphi = \varphi_1$ . Since  $F$  preserves the relation  $\approx$  by definition we obtain the following equivalence:

$$\begin{aligned} v_1 - v_2 = \eta(u_1, u_2) &\iff \\ \psi_1(v_1 - v_2) + \psi_2(u_1 - u_2) &= \eta(\varphi(u_1), \varphi(u_2)) + \eta(\varphi(u_1 - u_2), u_0) \end{aligned} \quad (28)$$

for all vectors  $v_1, v_2 \in V'$ ,  $u_1, u_2 \in V$ . Substituting in (28)  $u_2 = \theta$  and  $v_1 = v_2$  we arrive to the condition a). In particular, from a) we obtain  $\psi_2(u_1 - u_2) = \eta(\varphi(u_1 - u_2), u_0)$  for all  $u_1, u_2$  in  $\mathbb{V}$ . Thus, assuming a) from (28) we get b). Finally,  $F$  has form (26), as required.  $\square$

**PROPOSITION 2.27.** *The group of automorphisms of  $\approx$  is transitive.*

**PROOF.** It suffices to compute the orbit  $\mathcal{O}$  of the point  $[0, \theta]$  under the group of affine automorphisms of  $\approx$ . From 2.26, the orbit  $\mathcal{O}$  contains all the vectors  $[\psi_2(\theta) + v_0, \varphi(0) + u_0] = [v_0, u_0]$  with suitable maps  $\psi_2, \varphi$ . Considering  $\varphi = \text{id}$ ,  $\psi_2(u) = \eta(\varphi(u), u_0)$ ,  $\psi_1 = \text{id}$  we get a class of affine automorphisms of  $\approx$ : those defined by the formula

$$F([v, u]) = [v + \eta(u, u_0) + v_0, u + u_0]$$

with arbitrary fixed  $u_0, v_0$ . So, each point of  $\mathfrak{A}$  is in  $\mathcal{O}$ .  $\square$

Combining 2.27 and 2.24 we get a theorem, which is important in the context of foundations of geometry of affine semipolar spaces.

**THEOREM 2.28.** *Let  $\mathfrak{B}$  be the affine semipolar space determined by a semiform that meets assumptions  $(*)$  of 2.24. For each pair  $p, q$  of points of  $\mathfrak{B}$  such that  $p \sim q$  the set*

$$\bigcap \{ \{x : x \approx y\} : y \approx p, q \} \quad (29)$$

is the line of  $\mathfrak{B}$  through  $p, q$ . Consequently, the class of lines of  $\mathfrak{B}$  is definable in terms of the binary collinearity  $\approx$  of  $\mathfrak{B}$ .

PROOF. In view of 2.27 without loss of generality we can assume that  $p = [\mathbf{0}, \theta]$  and then 2.24 yields the claim directly.  $\square$

### 3 Symplectic affine polar spaces

#### 3.1 Automorphisms

Now, we return to the notation of Subsection 1.3.

In view of 1.4,  $\text{Aut}(\mathfrak{U}) \subset \text{Aut}(\mathfrak{A})$ . Moreover, in view of 2.2, as a particular instance of 2.26 we get the following characterization.

**PROPOSITION 3.1.** *Let  $f \in \text{AF}(\mathfrak{A})$  be a linear (affine) automorphism of  $\mathfrak{A}$ . The following conditions are equivalent.*

- (i)  *$f$  is an automorphism of  $\mathfrak{U}$ .*
- (ii) *The map  $f$  is given by the formula*

$$f([a, u]) = [\alpha a + v \circ u + b, \varphi(u) + w], \quad [a, u] \text{ is a vector of } \mathbb{V}, \quad (30)$$

where  $\alpha \neq 0$  and  $b$  are scalars,  $v, w$  are vectors of  $\mathbb{V}$ ,  $\circ$  stands for the ‘‘Cartesian’’ scalar product, and  $\varphi$  is a linear bijection of  $\mathbb{V}$  such that

- a)  $v \circ u = \eta(\varphi(u), w)$  for every vector  $u$  of  $\mathbb{V}$ , and
- b)  $\eta(\varphi(u_1), \varphi(u_2)) = \alpha \eta(u_1, u_2)$ , for all vectors  $u_1, u_2$  of  $\mathbb{V}$ .

**NOTE.** For a given map  $\varphi$ , the condition b) uniquely determines the parameter  $\alpha$ . Similarly, for given  $\varphi$  and  $w$ , the map  $u \mapsto \eta(\varphi(u), w)$  is a functional in  $\mathbb{V}^*$  and thus there is a vector  $v$  that satisfies a).

Note that the group  $\text{Aut}(\mathfrak{U})$  does not contain a transitive subgroup of translations:

**LEMMA 3.2.** *Let  $[b, w]$  be a vector of  $\mathbb{Y}$ . Then  $\tau_{[b, w]} \in \text{Aut}(\mathfrak{U})$  iff  $w = \theta$ .*

PROOF. A translation as above is a linear map of the form (30) with the ‘‘linear part’’ equal to the identity, in particular, with  $v = \theta$ . From a) we get  $\eta(\varphi(u), w) \equiv 0$  so,  $w = \theta$ .  $\square$

Similarly, from 3.1 we get

**LEMMA 3.3.** *Let  $\psi$  be a linear map of  $\mathbb{Y}$ . Clearly,  $\psi \in \text{Aut}(\mathfrak{A})$ . The following conditions are equivalent*



- (i)  $\psi \in \text{Aut}(\mathfrak{U})$ .
- (ii) There is a nonzero scalar  $\alpha$  and a linear bijection  $\varphi$  of  $\mathbb{V}$  such that
  - a)  $\psi([a, u]) = [\alpha a, \varphi(u)]$  for every vector  $[a, u]$  of  $\mathbb{Y}$ ;
  - b)  $\eta(\varphi(u_1), \varphi(u_2)) = \alpha \eta(u_1, u_2)$  for all vectors  $u_1, u_2$  of  $\mathbb{V}$  i.e.  $\varphi$  preserves  $\eta$ .

Nevertheless, as a direct consequence of 2.27 we infer that the structure  $\mathfrak{U}$  is homogeneous:

**PROPOSITION 3.4.** *The group  $\text{Aut}(\mathfrak{U}) \cap GL(\mathbb{Y})$  acts transitively on the point set of  $\mathfrak{U}$ .*

Let  $f$  have the form (30) such that a) and b) hold. Let us write  $\alpha = |\varphi|$ ; then b) assumes form  $\eta(\varphi(u_1), \varphi(u_2)) = |\varphi| \eta(u_1, u_2)$ . Clearly, the map  $\varphi \mapsto |\varphi|$  is a homomorphism of the group of "admissible"  $\varphi$ 's and the multiplicative group of  $\mathfrak{F}$ . With this notation, the formula (30) can be rewritten in the form

$$f([a, u]) = [|\varphi|a + \eta(\varphi(u), w) + b, \varphi(u) + w]. \quad (31)$$

Consequently, every  $f \in \text{Aut}(\mathfrak{U}) \cap GL(\mathbb{Y})$  can be identified with a triple  $(b, w, \varphi) \in F \times \mathbb{V} \times \text{Aut}(\eta)$ , where  $\text{Aut}(\eta)$  is the group of automorphisms of  $\perp_\eta$  on  $\mathbb{V}$ , i.e.

$$\text{Aut}(\eta) = \{\varphi \in GL(\mathbb{V}) : \exists \alpha \in F [\alpha \neq 0 \wedge \forall u_1, u_2 \eta(\varphi(u_1), \varphi(u_2)) = \alpha \eta(u_1, u_2)]\}.$$

Let  $f_i$  be associated with the triple  $(b_i, w_i, \varphi_i)$  for  $i = 1, 2, 3$  and let  $f_3 = f_2 f_1$ . First, we directly get  $\varphi_3 = \varphi_2 \varphi_1$ . From well known formula of (analytical) affine geometry  $(\tau_{\omega_2} \psi_2)(\tau_{\omega_1} \psi_1) = \tau_{\omega_2 + \psi_2(\omega_1)}(\psi_2 \psi_1)$ , where  $\psi_1, \psi_2$  are linear bijections we obtain  $w_3 = \varphi_2(w_1) + w_2$  and  $b_3 = |\varphi_2|b_1 + \eta(\varphi(w_1), w_2) + b_2$ . The obtained rules for transformation of parameters  $\varphi, w$  are analogous to the transformation rules of the direct product  $\text{Tr}(\mathbb{V}) \rtimes \text{Aut}(\eta)$ , but the transformation rule of  $b$  is more complex.

**LEMMA 3.5.** *Let  $\sigma \in \text{Aut}(\mathfrak{F})$ , let us fix a natural basis of  $\mathbb{V}$  and for  $u = [\alpha_1, \dots, \alpha_n]$  in  $\mathbb{V}$  let us set  $\sigma(u) = [\sigma(\alpha_1), \dots, \sigma(\alpha_n)]$ . Finally, we set  $\sigma^*([a, u]) = [\sigma(a), \sigma(u)]$  for  $[a, u]$  in  $\mathbb{Y}$ . Then  $\sigma^* \in \text{Aut}(\mathfrak{U})$ .*

PROOF. It suffices to note that for any vectors  $u_1, u_2$  in  $\mathbb{V}$  we have  $\eta(\sigma(u_1), \sigma(u_2)) = \sigma(\eta(u_1, u_2))$  and use 1.3.  $\square$

Summing up 3.1 and 3.5 we obtain a characterization of the group  $\text{Aut}(\mathfrak{U})$ .

**COROLLARY 3.6.** *The group  $\text{Aut}(\mathfrak{U})$  consists of all the maps*

$$\mathbb{Y} \ni [a, u] \mapsto [\alpha \sigma(a) + \eta(\varphi(u), w) + b, \varphi(a) + w]$$

*with a nonzero scalar  $\alpha$ , a scalar  $b$  in  $\mathfrak{F}$ , an automorphism  $\sigma$  of  $\mathfrak{F}$ , and a  $\sigma$ -semilinear bijection  $\varphi$  of  $\mathbb{V}$  such that  $\eta(\varphi(u_1), \varphi(u_2)) = \alpha \sigma(\eta(u_1, u_2))$  for  $u_1, u_2 \in \mathbb{V}$ .*

The natural question appears what is the "metric" geometry of our symplectic affine spaces i.e. what are characteristic relations defined on the point universe of  $\mathfrak{U}$  that characterize our geometry (except  $\sim$ , of course, which is sufficient, but is more affine than metric in spirit). It is clear that no relation that is invariant under all the translations can be used here. In particular, no line orthogonality can be used.

For pairs  $(p_1, p_2), (p_3, p_4)$  of points of  $\mathfrak{A}$  we define

$$p_1 p_2 \equiv p_3 p_4 : \iff \rho(p_1, p_2) = \rho(p_3, p_4). \quad (32)$$

Clearly, the relation  $\equiv$  is an equivalence relation. The following relation is crucial. For distinct  $p_1, p_2$  and arbitrary  $p$  we have

$$p_1 p_2 \equiv p p \text{ iff } p_1 \sim p_2.$$

Since  $\sim$  is expressible in terms of  $\equiv$ , each automorphism of  $\equiv$  preserves  $\sim$ , so, in view of 1.4, it is an automorphism of  $\mathfrak{A}$ . From 3.1, 1.4, and the above we can directly compute

**PROPOSITION 3.7.** *The following conditions are equivalent.*

- (i)  $f \in \text{Aut}(\langle Y, \equiv \rangle)$ .
- (ii)  $f \in \text{Aut}(\mathfrak{A})$ .
- (iii) *There is a nonzero scalar  $\alpha$  and an automorphism  $\sigma$  of  $\mathfrak{F}$  such that we have  $\rho(f(p_1), f(p_2)) = \alpha \sigma \rho(p_1, p_2)$  for all points  $p_1, p_2$  of  $\mathfrak{A}$ .*

### 3.2 Bisectors and symmetries

In this section we also follow notation of Subsection 1.3. Next, is a technical fact which we need later.

**LEMMA 3.8.** *For fixed  $w, \alpha, \beta$  the set*

$$\mathcal{Z} = \{[a, u] : \eta(w, u) = \beta + \alpha a\} \quad (33)$$

*either is empty ( $w = \theta, \alpha = 0, \beta \neq 0$ ), or it is the point set of  $\mathfrak{A}$  ( $w = \theta, \alpha, \beta = 0$ ), or it is a hyperplane of  $\mathfrak{A}$ .*

PROOF. From 2.21, the set  $\mathcal{Z}$  is a subspace of  $\mathfrak{A}$ . From 2.22, if  $w \neq \theta$  then  $\dim(\mathcal{Z}) = \dim(\mathfrak{A}) - 1$ , which is our claim.  $\square$

The relation  $\equiv$  defined by (32) has some properties of an abstract “equidistance relation” or a “segment congruence”, but note that it is not associated with any norm. From definition we have  $\rho(p_2, p_1) = -\rho(p_1, p_2)$ , which gives

$$p_1 p_2 \equiv p_3 p_4 \text{ iff } p_2 p_1 \equiv p_4 p_3, \text{ but } p_1 p_2 \equiv p_2 p_1 \text{ iff } p_1 \sim p_2.$$

Another similarity concerns bisector hyperplanes. In the context of a “metric” geometry determined by an equidistance relation  $\equiv$ , with a pair of points  $p_1, p_2$  one can associate “bisectors” of  $p_1, p_2$ :

$$\mathbf{L}_t \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \{p : p_1 p \equiv p_2 p\} \quad \text{and} \quad \mathbf{L}_m \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \{p : p_1 p \equiv p p_2\}. \quad (34)$$

(If in a geometry we have  $pq \equiv qp$  for all points  $p, q$ , then there is no need to distinguish between these two types of bisectors. In our case the distinction is necessary.) The following is evident.

**Remark 2.** Let  $p_0$  be a point of  $\mathfrak{A}$ . Then  $\mathbf{L}_t \begin{pmatrix} p_0 \\ p_0 \end{pmatrix}$  is the point set of  $\mathfrak{A}$  and  $\mathbf{L}_m \begin{pmatrix} p_0 \\ p_0 \end{pmatrix}$  is the hyperplane of  $\mathfrak{A}$  consisting of the points collinear in  $\mathfrak{A}$  with  $p_0$ .

Let  $p_i = [a_i, u_i]$  for  $i = 1, 2$ , directly from definitions (5) and (32) we compute the following

$$\begin{aligned}\mathbf{L}_t^{(p_1)} &= \{[a, u]: \eta(u_1 - u_2, u) = a_1 - a_2\} \text{ and} \\ \mathbf{L}_m^{(p_1)} &= \{[a, u]: \eta(u_1 + u_2, u) = (a_1 + a_2) - 2a\}.\end{aligned}$$

Thus by 3.8, we can complete 2 as follows.

**PROPOSITION 3.9.** *Let  $p_1, p_2$  be a pair of distinct points of  $\mathfrak{A}$ .  $\mathbf{L}_t^{(p_1)}$  is empty iff  $\mathbf{U}$  is the direction of  $\overline{p_1, p_2} =: L$ . When  $\mathbf{U}$  is not the direction of  $L$  then  $\mathbf{L}_t^{(p_1)}$ , and in any case  $\mathbf{L}_m^{(p_1)}$  both are hyperplanes of  $\mathfrak{A}$ .*

What is more intriguing is that an analogue of a “sphere”

$$\{p: p_1 p \equiv p_1 p_2\} = \{[a, u]: \eta(u_1, u) = (\eta(u_1, u_2) - a_2) + a\}$$

is a hyperplane of  $\mathfrak{A}$  as well.

Comparing the properties of  $\equiv$  defined here and of the equidistance of a metric affine space (characterized in [19]) we see that the only one axiom of [19] that is not valid here is this which states that opposite sides of a parallelogram are congruent (i.e. stating that a translation is an “isometry”). Some other (we believe: interesting) properties are valid, instead.

Let us write  $\oplus$  for the (affine) midpoint operation defined in  $\mathfrak{A}$ . Simple computation gives

**LEMMA 3.10.**  *$p_1 p_1 \oplus p_2 \equiv p_1 \oplus p_2 p_2$  for each pair  $p_1, p_2$  of points of  $\mathfrak{A}$ . Consequently,  $p_1 \oplus p_2 \in \mathbf{L}_m^{(p_1)}$ .*

Given a hyperplane  $H$  we say that  $p_1, p_2$  are *symmetric* wrt.  $H$  when  $H$  is the bisector hyperplane of  $p_1, p_2$ . In a metric affine geometry this notion can be used to define the axial symmetry. In our geometry we have, formally, two notions of a pair symmetric under a hyperplane.

**LEMMA 3.11.** *Let  $q = [b, w]$  be an arbitrary vector of  $\mathbb{Y}$ . Then*

$$\mathbf{L}_t^{(p_1)} = \mathbf{L}_t^{(p_2)} \text{ and } \mathbf{L}_m^{(p_1)} = \mathbf{L}_m^{(p_2)}$$

*for any pair  $p_1, p_2$  of points of  $\mathfrak{A}$ .*

PROOF. Let  $p_i = [a_i, u_i]$  and  $r = [c, y]$ . Clearly,  $\mathbf{L}_t^{(p_1)} = \mathbf{L}_t^{(p_2)}$  immediately follows from the condition

$$\rho(p_1 + q, r) - \rho(p_1, r) = \rho(p_2 + q, r) - \rho(p_2, r) \text{ for each } r.$$

Let us verify this condition. We compute:  $\rho(p_i + q, r) = \eta(u_i + w, y) - (a_i + b - c)$ ,  $\rho(p_i, r) = \eta(u_i, y) - (a_i - c)$ ,  $\rho(p_i + q, r) - \rho(p_i, r) = \eta(w, y) - b$ . This closes the proof of our first equality. The second one is computed analogously.  $\square$

Let us prove an auxiliary fact

**FACT 3.12.** *Let  $w, u, \alpha, \beta$  be fixed. Suppose that  $\eta(w, u) = \alpha \iff \eta(y, u) = \beta$  holds for all vectors  $u$ . Then  $y = \gamma w$  and  $\beta = \gamma \alpha$  for a nonzero scalar  $\gamma$ .*

PROOF. Take  $w', y'$  such that  $\eta(w, w') = \alpha$  and  $\eta(y, y') = \beta$ . From the assumptions we get  $\eta(w, u - w') = 0 \iff \eta(y, u - y') = 0$  i.e.  $u - w' \in w^\perp \iff u - y' \in y^\perp$ . This gives  $u' + w^\perp = y' + y^\perp$  from which  $w^\perp = y^\perp$  follows. Thus  $y \parallel w$ , so  $y = \gamma w$  for some  $\gamma$ . Finally,  $\beta = \gamma \alpha$  is directly computed.  $\square$

**LEMMA 3.13.** *Let  $p_i = [a_i, u_i]$  and  $p'_i = [a'_i, u'_i]$  for  $i = 1, 2$ . Then*

- (i)  $\mathbf{L}_t(p_2^{(p_1)}) = \mathbf{L}_t(p_2^{(p'_1)})$  *iff*  $(p_2 - p_1)\gamma = p'_2 - p'_1$  *for a nonzero scalar  $\gamma$ , and*
- (ii)  $\mathbf{L}_m(p_2^{(p_1)}) = \mathbf{L}_m(p_2^{(p'_1)})$  *iff*  $p_2 + p_1 = p'_2 + p'_1$ .

PROOF. Similarly as in 3.11 we compute that  $\mathbf{L}_t(p_2^{(p_1)}) = \mathbf{L}_t(p_2^{(p'_1)})$  is equivalent to the condition  $(\forall u)[\eta(u_1 - u_2, u) = (a_1 - a_2) \iff \eta(u'_1 - u'_2, u) = (a'_1 - a'_2)]$ . Applying 3.12 we get (i).

Analogously,  $\mathbf{L}_m(p_2^{(p_1)}) = \mathbf{L}_m(p_2^{(p'_1)})$  is equivalent to the condition

$$(\forall u)(\forall a)[\eta(u_1 + u_2, u) = (a_1 + a_2 - 2a) \iff \eta(u'_1 + u'_2, u) = (a'_1 + a'_2 - 2a)].$$

Substituting  $a = 0$  to the above we get  $u'_1 + u'_2 = (u_1 + u_2)\gamma$  for some  $\gamma \neq 0$  and  $a'_1 + a'_2 = (a_1 + a_2)\gamma$ . Considering again the above with  $a = 1$  we obtain  $\gamma = 1$ . This, together with 3.11 proves (ii).  $\square$

Note that from 3.13 in case (i), right-to-left with  $\gamma = 1$  we get 3.11.

The statements 3.11 and 3.13 can be summarized in the following

**THEOREM 3.14.** *Let  $H$  be a hyperplane of  $\mathfrak{A}$ . The relation  $p_1 \sigma_H^\dagger p_2$  defined by the condition  $\mathbf{L}_t(p_2^{(p_1)}) = H$  is not a function; nevertheless, whenever  $p_1 \sigma_H^\dagger p_2$  holds, the translation which maps  $p_1$  onto  $p_2$  is a subset of  $\sigma_H^\dagger$ .*

*The relation  $p_1 \sigma_H^m p_2$  defined by the condition  $\mathbf{L}_m(p_2^{(p_1)}) = H$  is either void or it is the central symmetry of  $\mathfrak{A}$  with the centre  $p_1 \oplus p_2$ .*

Finally, a result of a simple computation which has interesting consequences:

**PROPOSITION 3.15.** *Let  $p_i = [a_i, u_i]$  be distinct points of  $\mathfrak{A}$ ,  $q = p_1 \oplus p_2$ , and  $\vartheta$  be the direction of the line  $\overline{p_1, p_2}$  (i.e.  $q = [\frac{a_1+a_2}{2}, \frac{u_1+u_2}{2}] \in \mathbb{Y}$ ,  $\vartheta = [0, a_2 - a_1, u_2 - u_1] \in \mathbb{W}$ ). Then*

$$\mathbf{L}_m(p_2^{(p_1)}) = \{r : r \text{ is a point of } \mathfrak{A}, q \sim r\}, \quad \mathbf{L}_t(p_2^{(p_1)}) = \{r : r \text{ is a point of } \mathfrak{A}, \vartheta \perp_\xi r\}.$$

*Consequently,  $\mathbf{L}_m(p_2^{(p_1)})$  is the restriction of the hyperplane  $(p_1 \oplus p_2)^\perp$  of  $\mathfrak{P}$  polar of  $p_1 \oplus p_2$  to the point set of  $\mathfrak{A}$ . Analogously,  $\mathbf{L}_t(p_2^{(p_1)})$  is the restriction of the hyperplane which has the direction  $\overline{p_1, p_2}^\perp$  as its pole under the underlying null-polarity.*

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